Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

DECOMPOSITION INTO PARTIAL FRACTIONS BY MEANS OF REMAINDERS.

By Eugene Randolph Smith.

It has always seemed to the author of this paper that there should be some extension of the method of solving partial fractions by means of substitution, so that this method would be applicable to all cases without introducing any undue complexity. After several abortive attempts along the line of substitutions, the following method by the equivalent process of taking remainders was found.

There can be no claim that the method is entirely new, for other writers, for example, Chrystal, have used some of the same ideas, and have gotten some of the same results, though by somewhat different processes. The arrangement of the work in some of the cases is, as far as the author knows, an original one.

The proof for all cases is based upon the fact that two identical expressions, when divided by the same divisor, must give identical quotients and identical remainders. In certain questions this theorm is applied to several divisions taken successively.

The method is simple in application, presenting very little, even of arithmetical calculation, in ordinary fractions. The fractions having unfactorable (into rational factors) denominators present more algebraic difficulty than other types, but one of these of the nth degree requires only the solution of $(n-1)$ first degree equations in the same number of unknowns.

The method can be applied to find the numerator for any one factor of the denominator (except powers of the same quantity) independently of the other factors.

The following illustrations will show the application of the method to the various types. In the division used in this paper the "multiply and add" method of synthetic division will be used for first degree binomial divisors, and successive remainders in the numerator will be represented by $R1$, $R2$, etc., and in the denominator by $r1$, $r2$, etc.

To decompose into partial fractions:

$$\frac{x^5-x^4-25x^3+68x^2-13x-38}{(x-1)(x-2)^3(x^2+x+2)}.$$

I. To determine the numerator for a denominator of the type $ax+b$: in this case, to find the numerator A for the denominator $(x-1)$.

Divide the numerator, and all of the denominator except $(x-1)$, by the required denominator $(x-1)$. Call the remainders R_1 and r_1.

$$\begin{array}{r|l} & \text{Numerator.} \\ & 1-1-25+68-13-38 \\ 1 & 1\quad 0-25+43+30\,\big|\,\overline{-\ 8}=R_1 \end{array} \qquad \begin{array}{r|l r|l} & \text{Denominator.} & & \\ & 1-2 & & 1+1+2 \\ 1 & 1\,\big|\,\overline{-1} & 1 & 1+2\,\big|\,\overline{+4} \end{array}$$

Therefore $(-1)^3x(4)=-4=r_1$.

Then $r_1A=R_1$, that is, $A=\frac{-8}{-4}=2$, and the fraction is $\frac{2}{x-1}$.

In this type the method is evidently the same as the substitution method except that the results of substitution are obtained by division.

II. To determine the numerators for successive powers of $ax+b$: in this case the numerators A, B, and C for the denominators $(x-2)^3$, $(x-2)^2$, and $(x-2)$ respectively.

Multiply all the factors of the denominator of the given fraction except $(x-2)^3$, obtaining x^3+x-2.

Divide the numerator, and the product obtained in the denominator, by $(x-2)$, the quotient by $(x-2)$ again, and so on, until as many divisions have been performed (in this case three) as there are numerators to be obtained.

$$\begin{array}{r|l} & \text{Numerator.} \\ & 1-1-25+68-13-38 \\ 2 & 1+1-23+22+31\,\big|\,\overline{+24}=R_1 \\ & 1+3-17-12\,\big|\,\overline{+\ 7}=R_2 \\ & 1+5-\ 7\,\big|\,\overline{-26}=R_3 \end{array} \qquad \begin{array}{r|l} & \text{Denominator.} \\ & 1+0+\ 1-2 \\ 2 & 1+2+\ 5\,\big|\,\overline{+8}=r_1 \\ & 1+4\,\big|\,\overline{+13}=r_2 \\ & 1\,\big|\,\overline{+6}=r_3 \end{array}$$

Then

$$r_1A = R_1, \text{ which gives } A = \frac{R_1}{r_1} = 3,$$
$$r_1B + r_2A = R_2, \text{ which gives } B = -4,$$
$$r_1C + r_2B + r_3A = R_3, \text{ which gives } C = 1,$$

and the fractions are

$$\frac{3}{(x-2)^3} - \frac{4}{(x-2)^2} + \frac{1}{x-2}.$$

As r_1, r_2, etc., are coefficients of A, B, and C in order, these equations are easily remembered. There are several evident ways to write the six numbers with A, B, and C so as to find the results with a minimum of calculation. It is evident that each equation contains but one unknown, and that the first step of the method is identical to the method of I.

III. To determine the numerator for an unfactorable denominator of degree higher than the first: in this case the numerator $Ax + B$ for the denominator $(x^2 + x + 2)$.

Multiply together all the factors of the denominator except $(x^2 + x + 2)$, obtaining the product $x^4 - 7x^3 + 18x^2 - 20x + 8$.

Divide the numerator, and the product obtained in the denominator by $(x^2 + x + 2)$. After obtaining the first remainder in the denominator, multiply the dividend by x and divide again, obtaining another remainder. If there are three unknowns in the required numerator, the dividend must next be multiplied by x^2, and again divided, and so on, until as many divisions have been performed in the denominator as there are unknowns in the required numerator.

In practice, these divisions can all be done in one, as the first division is identical in all of them, and the annexing of a zero detached coefficient raises the degree by one. The successive remainders are not first, second and third remainders as in the previous case, so will be indicated by r, r', r'', etc.

Numerator.

$$\begin{array}{r|rrrrrrl}
 & 1 & -1 & -25 & +68 & -13 & -38 & \\
-2 & & & -2 & +4 & +50 & -194 & \\
-1 & & -1 & +2 & +25 & -97 & & \\
\hline
1 & 1 & -2 & -25 & +97 & \multicolumn{1}{|r}{-60} & -232 & R
\end{array}$$

$$\begin{array}{r|rrrrr|rrl}
 & \multicolumn{5}{c}{\text{Denominator.}} & \multicolumn{3}{c}{\text{Second Division.}} \\
 & 1 & -7 & +18 & -20 & +8 & +0 & & \\
-2 & & & -2 & +16 & -48 & +56 & & \\
-1 & & -1 & +8 & -24 & +28 & & & \\
\hline
1 & 1 & -8 & +24 & -28 & -40 & & & r \\
 & & & & & -12 & +56 & & r'
\end{array}$$

Equating the independent terms, the first degree terms, etc., of the remainders by the rule, $rB + r'A = R$, sufficient equations to determine the unknowns in the required numerator will be found. (For three unknowns, the rule would be $rC + r'B + r''A = R$.) In this case, $-40B + 56A = -232$, and $-28B - 12A = -60$, giving $A = -2$, and $B = 3$, and the fraction is

$$\frac{-2x + 3}{x^2 + x + 2}.$$

Certain special cases, particularly where a denominator is a perfect power, merit attention.

(1) To decompose $\dfrac{8x^2 - 18x + 14}{(2x - 3)^3}$.

Divide three times by $(2x - 3)$, and the successive remainders are the required numerators. If $x - (3/2)$ is used as the divisor, remainders after the first must be divided by successive powers of 2 to equal the remainders for $(2x - 3)$.

$$\begin{array}{r|l}
 & 8 - 18 + 14 \\
3/2 & 8 - 6 \,|+ 5 = A \\
 & 8 \,|+ 6 = 2B \\
 & 8 = 4C
\end{array}$$

The numerators are 5, 6/2 or 3, and 8/4 or 2, giving

$$\frac{5}{(2x - 3)^3} + \frac{3}{(2x - 3)^2} + \frac{2}{(2x - 3)}.$$

(2) To decompose $\dfrac{3x^3 - 2x^2 + 10x}{(x^2 - x + 2)^2}$. Divide by $(x^2 - x + 2)$

$$\begin{array}{r|cccl}
 & 3 & -2 & +10 & +0 \\
-2 & & & -6 & -2 \\
1 & & 3 & 1 & \\
\hline
1 & 3 & +1 & +5 & -2 = R_1.
\end{array}$$

The first remainder is evidently $5x-2$, and the second remainder is the same as the quotient, or $3x+1$, giving

$$\frac{3x+1}{x^2-x+2}+\frac{5x-2}{(x^2-x+2)^2}.$$

(3) To decompose $\dfrac{6x^4-14x^3+29x^2-32x+12}{(x^2-x+2)^2(x-2)}$.

This requires a combination of methods II and III in finding the numerators for the powers of (x^2-x+2). To find the numerator $Ax+B$ for (x^2-x+2), and $Cx+D$ for $(x^2-x+2)^2$, divide twice by (x^2-x+2).

Numerator.

$$\begin{array}{r|ccccl}
 & 6 & -14 & +29 & -32 & +12 \\
-2 & & & -12 & +16 & -18 \\
1 & & +6 & -8 & +9 & \\
\hline
1 & 6 & -8 & +9 & -7 & -6 = R_1 \\
\hline
-2 & & & -12 & & \\
1 & & 6 & & & \\
\hline
1 & 6 & -2 & -3 = R_2 & &
\end{array}$$

Denominator.

$$\begin{array}{ccc}
1 & -2 & r_1 \\
0 & 0 & r_2
\end{array}$$

Denominator multiplied by x.

$$\begin{array}{r|cccl}
 & 1 & -2 & +0 & \\
-2 & & & -2 & \\
1 & & 1 & & \\
\hline
1 & 1 & -1 & -2 & r_1'. \\
 & 1 & & & r_2'
\end{array}$$

Then, using r_1 with B and r_1' with A, $B-A=-7$ and $-2B-2A=-6$, and $A=5$, $B=-2$.

Also, using r_1 with D, r_1' with C, r_2 with B and r_2' with A, $A-2C-2D=-3$, $-C+D=-2$, giving, since $A=5$, $C=3$ and $D=1$.

(4) To decompose a fraction whose numerator is of higher degree than its denominator.

The method for any single partial fraction is exactly the same as in the cases shown. If the fraction is to be completely decomposed, the method will accomplish this automatically if the denominator is a perfect power; otherwise it is best to divide the numerator by the denominator as the first step.

THE PARK SCHOOL,
BALTIMORE, MD.